Explosive Roots in Level Vector Autoregressive Models*

Hammad Qureshi[†]

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Abstract

Level vector autoregressive (VAR) models are used extensively in empirical macroeconomic research. However, estimated level VAR models may contain explosive roots, which is at odds with the widespread consensus among macroeconomists that roots are at most unity. This paper investigates the frequency of explosive roots in estimated level VAR models in the presence of stationary and nonstationary variables. Monte Carlo simulations based on datasets from the macroeconomic literature reveal that the frequency of explosive roots exceeds 40% in the presence of unit roots. Even when all the variables are stationary, the frequency of explosive roots is substantial. Furthermore, explosion increases significantly, to as much as 100% when the estimated level VAR coefficients are corrected for small-sample bias. These results suggest that researchers estimating level VAR models on macroeconomic datasets encounter explosive roots, a phenomenon that is contrary to common macroeconomic belief, with a very high frequency. Monte Carlo simulations in the paper reveal that imposing unit roots in the estimation can substantially reduce the frequency of explosion. Hence one way to mitigate explosive roots is to estimate vector error correction models.

Keywords: Explosive Roots, Level VAR Models, Bias Correction, VECMs

JEL Classification: E32, C32

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[†]Department of Economics, The Ohio State University, 410 APRS Hall, 1945 N. High Street, Columbus, OH 43210. *E-mail address*: qureshi.18@osu.edu.

1. Introduction

Following the work of Sims (1980), impulse response analysis based on level vector autoregressive (VAR) models has been utilized in numerous studies and plays an important role in contemporary macroeconomic research.[1] However, estimated level VAR models may contain explosive roots even if all the true autoregressive roots lie inside the unit circle. The incidence of such explosive roots is at odds with the widespread agreement among macroeconomists that roots are at most unity.[2] Given that level VAR models are used extensively and may estimate roots greater than unity, it is important to examine how frequently researchers estimating level VAR models on macroeconomic datasets encounter explosive roots.

This paper investigates this frequency using Monte Carlo simulations based on datasets that are representative of those commonly used in the macroeconomic literature. In specific, datasets from three highly cited papers in the literature, Christiano, Eichenbaum, & Evans (1999, 2005), CEE henceforth, and Eichenbaum & Evans (1995), EE henceforth, are employed to examine the frequency of explosive roots (explosion) in estimated level VAR models.[3] Monte Carlo samples are generated under two specifications of the data-generating process (DGP). The first specification of the DGP imposes unit roots in the simulated data, while the second specification is based on a stationary process. Subsequently, level VAR models are estimated on the simulated data to compute the frequency of explosive roots. Under both these specifications, this paper also examines the frequency of explosion after correcting for the small-sample bias in estimated level VAR coefficients.

Monte Carlo results in this study reveal that the frequency of explosive roots exceeds

[1] One advantage of level VAR models over alternatives such as the vector error correction models is that the former are robust to the number of unit roots in the system. This robustness is one of the reasons why level VAR models are used extensively in applied macroeconomic research.

[2] Macroeconomists may model few phenomenon such as hyperinflations as explosive processes (see Neilsen (2005) and Juselius (2002)). These are important but very specific cases and in general most macroeconomic variables are modeled as non-explosive processes.

[3] CEE (1999), CEE (2005) and EE (1995) are among the most highly cited papers in the applied macroeconomic research. Differences in VAR order, data frequency and variables used in these papers facilitate the assessment of explosive roots under a variety of specifications.

40% in the presence of unit roots. Even when all the variables are stationary, the frequency of explosive roots is substantial; it is as high as 25%. Furthermore, explosion increases significantly, to more than 90% under several specifications, when the estimated level VAR coefficients are corrected for small-sample bias. These results suggest that researchers estimating level VAR models on macroeconomic datasets encounter explosive roots, a phenomenon that is contrary to common macroeconomic belief, with a very high frequency.

Considering the consensus among macroeconomists that roots are at most unity, applied macroeconomists may discard explosive VAR draws in *simulated* data used for constructing confidence intervals for the impulse responses.[4] However, discarding explosive VAR specifications when estimating level VAR model on the *actual* dataset is problematic because it may lead to biases in the estimation or even result in data mining. Data mining can be a serious problem since it invalidates statistical theory. The high frequency of encountering explosive roots in estimated level VAR models suggests that this data mining problem can be severe. Additionally, the sharp increase in explosion after bias correction in estimated level VAR coefficients indicates that researchers correcting for the small-sample bias in these coefficients may encounter explosive roots with an even higher probability.

As per the well known evidence of nonstationarity in most macroeconomic series, one way to reduce the frequency of explosive roots is to impose unit roots in the estimation by estimating VECMs instead of level VAR models. I examine the frequency of explosive roots in estimated VECMs under the same specifications of the DGPs. Monte Carlo simulations reveal that explosion occurs much less frequently in estimated VECMs.

The rest of the paper is organized as follows. Section II examines the frequency of explosive roots in estimated level VAR models in the presence of nonstationary variables. Section III focuses on explosive roots in estimated level VAR models when all the variables are sta-

[4]For instance, Ditmar, Gavin & Kyland (2005) and Altig, Christiano, Eichenbaum & Linde (2004) discard explosive VAR draws in the *simulated* data used for constructing error bands for their impulse responses.

tionary. Section IV examines the frequency of explosive roots in estimated VECMs. Section V concludes.

2. Data Generating Process with NonStationary Variables

Many macroeconomists model highly persistent time series, such as inflation, interest rate, exchange rate and money demand, as unit root processes since empirical studies that estimate these series have mostly failed to reject the null hypothesis of unit root nonstationarity. Given this evidence for nonstationarity of several macroeconomic variables and that macroeconomic theory predicts that some of these series have long-run equilibrium relationships, this paper tests for unit roots and cointegration in CEE (1995), CEE (2005) and EE (1995) datasets.[5] Several unit root tests are implemented to test stationarity of macroeconomic variables in CEE (1999), CEE(2005) and EE (1995). These tests fail to reject the null of unit root for most macroeconomics series. Johansen's (1988) tests are used to estimate the cointegration ranks in the datasets. Based on these tests, cointegration ranks of five, four and two are used for the DGPs based on CEE (1999), CEE (2005) and EE (1995) respectively.[6] However, Podivinsky's (1998) results suggest that Johansen's cointegration test may not be very reliable, especially in shorter samples due to severe size distortions.[7] I therefore examine the sensitivity of the results to varying cointegration ranks in the DGPs.

Given the evidence for the existence of nonstationarity and cointegration, common stochastic trends are imposed in the Monte Carlo samples by estimating vector error correction models (VECMs) on the datasets and using the estimated regression coefficients for the DGP. Subsequently, the frequency of explosive roots in estimated level VAR models is computed.[8]

[5] Unit root tests, Cointegration tests and a description of the datasets are reported in the appendix.

[6] These cointegration ranks are chosen based on trace tests. The maximum eigenvalue tests, on the other hand, yield cointegration ranks of four for CEE (1999), and one for CEE (2005) and EE (1995). Given these mixed results, sensitivity of results to different cointegration ranks is also examined.

[7] Johansen(2002) proposes a small sample Barlett correction that improves the finite-sample performance of his test. However, Juselius(2006) points out that these corrections do not solve the power problem and in some cases the size of the test and the power of alternative hypotheses close to the unit circle are almost of the same magnitude.

[8] Unrestricted level VAR models are robust to the number of unit roots in the system and hence are not

Estimation Procedure

Monte Carlo experiments in the paper can be summarized into the following steps:

1. First I estimate reduced form VECMs using Johansen's maximum likelihood method on the datasets.[9]

   $$\Delta Y_t = c + \zeta_1 \Delta Y_{t-1} + \zeta_2 \Delta Y_{t-2} + ... + \zeta_{p-1} \Delta Y_{t-p+1} + \zeta_0 Y_{t-1} + \epsilon_t \tag{1}$$

   Assuming normal errors, VECM coefficients can be estimated by maximizing the following likelihood function:

   $$\begin{aligned} L(\Omega, \zeta_1, ..., \zeta_{p-1}, c, \zeta_0) = & (-Tn/2)\, log\,(2\pi) - (T/2)\, log\, |\Omega| \\ & - \frac{1}{2}\sum_{t=1}^{T} [(\Delta Y_t - c - \zeta_1 \Delta Y_{t-1} - ... - \zeta_{p-1} \Delta Y_{t-p+1} - \zeta_0 Y_{t-1})' \\ & \Omega^{-1} (\Delta Y_t - c - \zeta_1 \Delta Y_{t-1} - ... - \zeta_{p-1} \Delta Y_{t-p+1} - \zeta_0 Y_{t-1})] \end{aligned} \tag{2}$$

   subject to $\zeta_0 = -BA'$
   where Y_t is a n-dimensional vector of variables, Ω is the covariance matrix of ϵ_t, B is an $(n \text{ x } h)$ matrix, A' is an $(h \text{ x } n)$ matrix of cointegrating vectors, and h is the cointegration rank based on Johansen's test.[10]

2. Next I use the estimated VECM coefficients to generate 10,000 Monte Carlo samples.[11]

3. Finally I estimate level VAR models on each of these samples to get the reduced form coefficients:

   $$Y_t = c^i + \theta_1^i Y_{t-1} + \theta_2^i Y_{t-2} + ... + \theta_p^i Y_{t-p} + \epsilon_t \qquad \textit{for } i\text{=1,2,... 10,000} \tag{3}$$

   and subsequently check their stability to compute the frequency of explosive roots.[12]

misspecified in the presence of unit roots and cointegration, as is the case in the simulated data under this specification. However, estimating VAR in levels in the presence of cointegration involves a loss of efficiency because some restrictions, namely the reduced rank of ζ_0 in (2), are not imposed.

[9] VECM(4) is estimated on the CEE (1999) and CEE (2005) datasets, and VECM(6) is estimated on the EE (1995) dataset, since CEE (1999, 2005) used level VAR(4) and EE (1995) used VAR(6) specifications for their reduced form estimations.

[10] The likelihood function is maximized by implementing the step by step procedure proposed by Johansen (1988, 1991) as outlined in Hamilton (1994).

[11] Initial values from the datasets are used as the starting values for the Monte Carlo samples.

[12] Stability of a VAR(p) model can be checked by calculating λ_{max}, the modulus of the largest root of its companion matrix. If λ_{max} of an estimated VAR model lies outside the unit circle in a given sample, the VAR model would be unstable for that Monte Carlo sample. Frequency of explosive roots corresponds to the proportion of unstable VAR draws in the Monte Carlo samples. In order to allow for rounding off errors, I consider a VAR model to be explosive only if its λ_{max} exceeds a threshold value of 1.00001 (instead of exactly one). Results are essentially the same for other thresholds such as 1.0001 or 1.0005.

Theoretical Predictions

Consider an estimator that yields median-unbiased estimates of autoregressive roots in multivariate time series models. I refer to this imaginary estimator as 'median unbiased autoregressive roots estimator' (MUAR).[13] If the true data-generating process is a VECM and the magnitude of the largest autoregressive root, λ_{max}, is exactly one, we would expect to encounter explosive roots with a probability of 0.5 with MUAR. Needless to say, a 50% likelihood of explosion is extremely high. However, since the least-squares estimator used for level VAR estimations is not median-unbiased, we can expect lower frequency of explosive roots if the least-squares bias in λ_{max} is downward, whereas in the case of an upward bias it would be even higher.

Least-squares bias in autoregressive roots can be downward or upward. Andrews (1993) shows that the least-squares estimator is significantly downward biased in AR(1)/unit root models. Similarly Andrews and Chen (1994) show that least-squares estimates of α, the sum of autoregressive coefficients in AR(p) models, are substantially downward biased in small samples. However, since the mapping from autoregressive coefficients to autoregressive roots is nonlinear, the bias in autoregressive roots can go either way even if the autoregressive coefficients are downward biased. For instance, Andrews and Chen (1994, Table 2) report upward least-squares bias in most autoregressive roots.[14] Given that the least-squares estimator can be significantly biased in small samples and the bias in autoregressive roots can go in either direction, it is hard to predict how often estimated level VAR models may contain explosive roots. Consequently, Monte Carlo simulations are used to estimate the frequency of explosive roots in estimated level VAR models with and without correcting for

[13] It must be emphasized that no such estimator exists and MUAR is just an imaginary estimator, mentioned solely for expository purpose.

[14] Andrews and Chen (1994, Table 2) report results for three autoregressive models, which have upward bias in the magnitude of most autoregressive roots other than that of the largest one. Monte Carlo simulations (available upon request) based on their models with slightly different coefficient values yield upward bias in the magnitude of the largest root.

the small-sample bias using standard bias correction procedures.

Andrews (1993) and Andrews & Chen (1994) among others have proposed bias-corrected estimators for univariate autoregressive models. Kilian (1998) proposes a bias correction approach for multivariate time series models such as VAR. His approach relies on calculating the mean-bias using nonparametric bootstrapping. Nicholls and Pope (1988), on the other hand, provide a closed-form expression for the bias in stationary multivariate Gaussian autoregressions. Pope (1990) extends these results by relaxing the assumption of Gaussian innovations. It must be emphasized that common bias correction procedures, including those by Kilian (1998) and Pope (1990), are designed to correct the small-sample bias in the autoregressive *coefficients*, which may not correct the bias in autoregressive *roots* due to the nonlinear relationship between the two.[15] This paper uses bias correction procedures based on Kilian (1998) and Pope (1990) to correct for the small-sample bias in estimated level VAR coefficients.

Kilian's bias correction procedure involves estimating VAR models and generating N replications of the estimated coefficients using standard nonparametric bootstrap techniques. Subsequently, the mean-bias is estimated as the difference between the average of the N replications of coefficients and the initial estimate of coefficients used in the DGP. This procedure is computationally demanding since it requires generating N replications on each Monte Carlo sample. Therefore, this paper uses a modest number of Monte Carlo samples: it generates 1000 Monte Carlo samples for the bias correction simulations and estimates the bias using 1000 replications of the estimated coefficients on each Monte Carlo sample.[16]

[15] Median-unbiasedness is preserved under monotone transformation whereas mean-unbiasedness is presevered under linear combinations. Since the autoregressive roots are neither a monotone transformation nor a linear combination of the autoregressive coefficients, median or mean bias corrections in coefficients will not in general correct the bias in the roots. To my knowledge, there does not exist any bias correction procedure that is designed to correct the bias in autoregressive roots of VAR models. Qureshi (2008) proposes a method to numerically correct the median-bias in autoregressive roots.

[16] This would result in 1000^2 or one million simulations which take considerable time even with the fast processors available to date. Sensitivity of results to increasing the number of simulations to 2000^2 is examined for simulations reported in Table 1. Frequency of explosive roots essentially remains the same.

Kilian implements a stationarity correction after correcting the bias in coefficients to avoid pushing stationary impulse response estimates into the nonstationary region. Kilian's bias correction with stationarity correction would ensure that explosive roots in estimated VAR models are eliminated. However, Sims and Zha (1995) criticize Kilian's stationarity correction as '*ad hoc*'. This paper implements Kilian's bias correction method without the stationarity correction. Hence results in this paper reveal how frequently Kilian's method relies on stationarity correction to avoid explosive roots in estimated level VAR models.

Pope's expression for the mean-bias in VAR coefficients is defined for demeaned stationary VAR(1) models. In order to implement bias correction based on this expression, VAR(p)s are estimated on demeaned simulated data and then reformulated as VAR(1)s.[17] Subsequently, the mean-bias is calculated using Pope's expression. Finally the mean-bias is subtracted from the estimated VAR coefficients to yield bias-corrected coefficients. I refer to these steps as Pope's bias correction.[18]

Results

Figure 1 presents an example illustrating the frequency of explosive roots in estimated level VAR models. It plots the distribution of λ_{max}, the modulus of the largest autoregressive root, in estimated level VAR(4) models with and without bias correction. The DGP is based on VECM estimation on the CEE (1999) dataset, with a cointegration rank of five. The frequency of explosive roots corresponds to the area under the distribution to the right of

[17] Demeaned data for simulations with Pope's bias correction is generated by using estimated VECM coefficients in (1) without the constant, and by setting the initial values in the Monte Carlo samples to zero.

[18] In this paper the bias correction procedures are implemented only on stable VAR draws and explosive roots in unstable VARs are counted towards the frequency of explosion without bias correction. This is because Pope's solution for the bias in VAR coefficients is defined for stationary VAR models. Similarly, Kilian's approach is designed for stationary models. However, Kilian (1998) argues that based on the continuity of the finite-sample distribution of the OLS estimator, the bootstrap approximation may still be used for slightly explosive cases. In light of this argument, I estimate the frequency of explosive roots after implementing Kilian's and Pope's bias corrections on all Monte Carlo samples (including the explosive ones). Results based on this exercise are essentially the same as the benchmark case of bias correction on stable VARs only.

unity. The following tables report these frequencies under various specifications.[19]

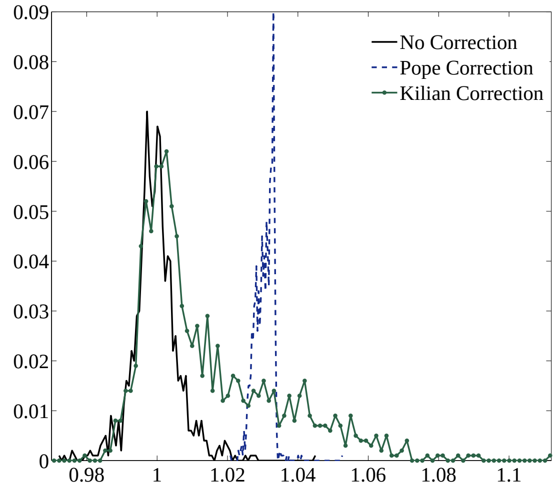

Figure 1: Distribution of λ_{max}

Table I reports the frequency of explosive roots in estimated level VAR models for the benchmark estimations of VAR(4) in CEE(1999) and CEE(2005), and VAR(6) in EE(1995). The frequency of explosion is considerably high. Estimated level VAR models have explosive roots in 46.4%, 47.7% and 41.9% of the Monte Carlo samples based on CEE (1999), CEE (2005) and EE (1995) respectively. Furthermore, the frequency of explosion increases substantially after correcting for the small-sample bias. Results for both Pope (1990) and Kilian (1998) bias correction procedures, denoted by 'Pope' and 'Kilian' respectively, are reported. Estimated level VAR models have explosive roots more that 75% of the time after Kilian's bias correction and 100% of the time after Pope's bias correction. Table 1 reveals that λ_{max} has downward median-bias because the frequency of explosive roots is less than 50% in the benchmark specifications. Additionally, Kilian's and Pope's bias corrections on level VAR coefficients overcorrect this bias in λ_{max}, consequently resulting in upward median-bias.

[19]EE (1995) estimate level VAR models with five, seven and eight variables and examine five different nominal and real exchange rates. For more details, please refer to the appendix. In the interest of brevity, this paper only presents results for their nominal \$/Franc exchange rate model with five variables. Results for other specifications and exchange rates are essentially the same. CEE (1999) report results with both $M1$ and $M2$ in their benchmark specification. This paper only presents results with $M1$. Once again, results are almost the same if $M1$ is replaced by $M2$.

These results suggest that researchers estimating level VAR on macroeconomic datasets, which include some nonstationary I(1) variables, encounter explosive roots very frequently, and even more so if they correct for the finite-sample bias in their estimation.

The following subsection examines the sensitivity of these results to varying cointegration ranks in the DGP, and to shorter samples and different lag orders in the estimated VAR models. Results from table 1 are reproduced (*in italics*) in the following tables to facilitate comparison with these *benchmark* specifications.

Sensitivity Analysis

Considering that cointegration rank tests may not be reliable in small samples, table 2 examines the sensitivity of results to different cointegration ranks in the DGP.[20] In most cases as the cointegration rank, h, increases, and hence the number of unit roots in the DGP decreases, the frequency of explosion goes down. For instance, as h increases from 3 to 6 in CEE (1999), the frequency of explosion decreases from 41.7% to 36.7%. However, explosion still remains high; in most simulations estimated level VAR models have explosive roots in more than 40% of the Monte Carlo samples. Once again, explosion increases substantially, to more than 90% in several cases, after bias correction. These results confirm that the high frequency of explosive roots is robust to varying cointegration rank in the DGP. The next table assesses the sensitivity of results to different subsamples and lag orders in estimated level VAR models.

Macroeconomic datasets for the post-Bretton Woods or post-Volcker eras are relatively short, which may exacerbate explosion in estimated level VAR models.[21] Level VAR models are estimated on truncated Monte Carlo samples, namely 'post-Bretton Woods' and 'Volcker

[20] Number of variables in the system, n, equals 7, 9 and 5 for CEE (1999), CEE (2005) and EE (1995) respectively. Any remaining cointegration ranks that are not reported yield very similar results.

[21] For example, if λ_{max} is biased downward explosion may rise due to an increase in the variance of λ_{max} in shorter samples. However, it must be emphasized that median-bias as well as other characteristics of the distribution (skewness, kurtosis, etc.) would also in general change in smaller samples making it hard to predict how the frequency of explosion would be affected.

& post-Volcker', to estimate the frequency of explosion in shorter samples.[22]

Different lag orders in estimated level VAR models may also affect the frequency of explosive roots. I therefore examine the sensitivity of results to varying orders in level VAR models in the full-sample as well as the two subsamples. CEE (1999) and CEE (2005) use level VAR(4) models while EE (1995) use level VAR(6) model for their reduced form estimation. Since these subsamples are fairly short, degrees of freedom would be low for the benchmark specifications of for four lags in CEE (1999, 2005) and six lags in EE (1995). Hence, the frequency of explosion is also reported for lower lag orders in level VAR models.

Table 3 reveals that the frequency of explosive roots remains high for different lag orders in estimated models. Moreover, explosion increases further in shorter samples in several simulations. For instance, the frequency of explosion in the benchmark cases increases to 56.1%, 70.2% and 44.9% in the 'Volcker & post-Volcker' subsamples. Once more, explosion increases appreciably after correcting for the small-sample bias in estimated level VAR coefficients. The frequency of explosive roots is more than 75% under all specifications after Kilian's bias correction and increases to 100% in all cases after Pope's correction.

3. Data Generating Process with Stationary Variables

The previous section examined the frequency of explosive roots in estimated level VAR models in the presence of unit root nonstationary variables. This section focuses on explosive roots in estimated level VAR models when all the variables are stationary. In this case the data-generating processes are based on level VAR models, as opposed to VECMs.

Estimation Procedure

The procedure for conducting Monte Carlo experiments is the same as that in the previous

[22] 'post-Bretton Woods' and 'Volcker & post-Volcker' subsamples correspond to the following sample periods: *CEE (1999, 2005)* quarterly - 'post-Bretton Woods' (1974:1 to 1995:2) and 'Volcker & post-Volcker' (1979:3 to 1995:2). *EE (1995)* monthly - 'post-Bretton Woods' (1974:1 to 1991:12) and 'Volcker & post-Volcker' (1979:8 to 1991:12).

section except for the first two steps in which level VAR(p) models are estimated on demeaned datasets and the corresponding coefficients are used to generate 10,000 Monte Carlo samples.[23] Starting values for the Monte Carlo samples are drawn from the stable VAR distribution. Subsequently, level VAR models are estimated on these samples to compute the frequency of explosive roots.[24]

Results

Table 4 summarizes results for the frequency of explosive roots in estimated level VAR models when the DGP is stationary. It presents results under the same specifications of estimated level VAR models as those reported in the table 3. Results in table 4 reveal that even in the absence of any unit roots, the frequency of explosive roots is considerable. Estimated level VAR models on full-samples have explosive roots in 25.9%, 12.9% and 19.6% of the simulations based on the benchmark specifications in CEE (1999), CEE (2005) and EE (1995) respectively. Furthermore, explosion increases substantially in shorter subsamples. For instance, the frequency of explosive roots in these benchmark cases increases to 56.0%, 61.4% and 31.7% respectively in the 'Volcker & post-Volcker' subsamples. Results for different lag orders in estimated models show that the high frequency of explosive roots is robust to varying order in level VAR estimation. As before, explosion increases substantially after bias correction. In most cases, explosive roots are encountered in more that 70% simulations after Kilian's bias correction and in more than 90% simulations after Pope's bias correction.

These results indicate that macroeconomists estimating level VAR model on datasets encounter explosive roots very frequently even if all the variables in their dataset are stationary. Moreover, they may almost always estimate explosive roots on macroeconomic datasets if they correct for the small-sample bias in level VAR coefficients.

[23] p equals 4 for the DGP based on CEE (1999, 2005) datasets and 6 for EE (1995) dataset. Since the DGP is based on demeaned data, I estimate level VAR models (without constant) on the Monte Carlo samples. This is useful for Pope's bias correction since Pope's expression is defined for demeaned stationary VARs.

[24] Since λ_{max} in the stationary DGP is less than unity, it is hard to predict the frequency of explosion even for the imaginary MUAR estimator.

Considering that the frequency of explosive roots in estimated level VAR models is very high, the next section explores alternatives to level VAR models and examines the frequency of explosion in one such alternative, namely the vector error correction models.

4. Explosive Roots in Vector Error Correction Models

As per the well known evidence of nonstationarity in most macroeconomic series, one way to reduce the frequency of explosive roots is to impose unit roots in the estimation by estimating VECMs instead of level VAR models.[25] This section examines the frequency of explosive roots in estimated VECMs under the same specifications of the DGPs as the previous sections.

Estimation Procedure

The procedure for conducting Monte Carlo experiments is identical to that in the previous sections, except for the third step in which VECMs are estimated on the simulated datasets instead of level VAR models. Cointegration ranks of five, four and two are imposed on each simulated dataset for CEE (1999), CEE (2005) and EE (1995) respectively. These cointegration ranks are the same as those imposed in the nonstationary DGP.[26]

Results

Table 5 reports the frequency of explosive roots in estimated VECMs when the DGP is nonstationary. Since the standard bias correction procedures for level VAR models can not be applied to VECMs, the following tables only report the cases without bias correction. These results reveal that the frequency of explosion reduces dramatically once VECMs are estimated on the simulated datasets. Imposing unit roots in the estimation restricts the magnitude of some of the explosive roots to unity, hence reducing the frequency of explo-

[25] Imposing unit roots in the estimation would restrict the magnitude of some of the otherwise explosive roots to unity, hence reducing the frequency of explosive roots.

[26] An alternative approach would be to estimate cointergration rank for each simulated dataset and impose the corresponding number of unit roots in the estimation.

sion. Based on the benchmark specifications in CEE (1999), CEE (2005) and EE (1995), estimated VECMs on full-samples have explosive roots in only 2.3%, 0.6% and 0.4% of the simulations respectively, compared to 46.4%, 47.7% and 41.9% in estimated level VAR models. Frequency of explosive roots in estimated VECMs increases to some extent in shorter subsamples. Explosion increases to 9.4% in the 'Volker & post-Volker' subsamples in the benchmark specification of CEE (1999). However, the frequency of explosive roots in estimated VECMs is still much lower than estimated level VAR models.

Table 6 reports the frequency of explosive roots when the DGP is stationary. It presents results under the same specifications of estimated VECMs as those reported in table 5. Once again, the frequency of explosive roots is very low. It is less that 1% under all specifications on full-samples. Estimated VECMs on full-samples have explosive roots in only 0.9%, 0.0% and 0.1% of the simulations based on the benchmark specifications in CEE (1999), CEE (2005) and EE (1995) respectively. Even in shorter samples, the frequency of explosive is less than 5% under most specifications.

The last table presents the sensitivity of these results to varying cointegration ranks in estimated VECMs. Results in table 7 reveal that the frequency of explosion decreases as more unit roots are imposed in the VECM estimation. For instance, explosion decreases from 47.7% to 5.1% in CEE (2005) simulations if the cointegration rank, h, is reduced from 9 to 7. These results show that the frequency of explosion in estimated VAR models can be reduced dramatically by imposing just one or two unit roots. This suggests that even if researchers are not confident about the exact cointegration rank, but expect atleast one or two unit roots in the system, they would be better off estimating VECMs with h equal to n-1 or n-2 as opposed to estimating a level VAR model (i.e. h equals n), where n in the number of variables in the system.

Overall, these results show that the frequency of explosive roots can be reduced substantially by estimating VECMs instead of level VAR models.

4. Conclusion

Level VAR models are used extensively in applied macroeconomic research. However, estimating VAR in levels may result in explosive roots even if all the true roots lie strictly inside the unit circle. The occurence of such explosive roots is inconsistent with the prevalent agreement among macroeconomists that roots are at most unity. Given that level VAR models are used extensively and may estimate roots greater than unity, this paper examines how frequently researchers estimating level VAR models on macroeconomic datasets may encounter explosive roots. Monte Carlo simulations based on datasets from the macroeconomic literature reveal that the frequency of explosive roots exceeds 40% in the presence of unit roots and is substantial even if all the variables are stationary. Furthermore, explosion increases substantially, to as much as 100%, after correcting for the small-sample bias in estimated level VAR coefficients.

These results suggest that researchers estimating level VAR models on macroeconomic datasets encounter explosive roots with a very high frequency. Considering the consensus among macroeconomists that roots are at most unity, if applied macroeconomists discard explosive VAR specifications when VAR models are estimated on the datasets, it may lead to biases in the estimation or can even result in data mining. Data mining can be a serious problem since it invalidates statistical theory. The high frequency of encountering explosive roots in level VAR models suggests that this data mining problem can be severe. Additionally, the sharp increase in explosion after bias correction indicates that researchers, who correct for the small-sample bias in level VAR coefficients, may almost always estimate explosive roots on macroeconomic datasets.

As per the well known evidence of nonstationarity in most macroeconomic series, one way to reduce the frequency of explosive roots is to impose unit roots in the estimation by estimating VECMs instead of level VAR models. Simulation results suggest that VECMs can substantially reduce the frequency of explosive roots. Another alternative could be imposing cointegrating relationships among variables in the VAR as in Shapiro & Watson (1988).[27] Evaluating these alternatives in terms of the accuracy of estimated impulse responses, variance decompositions and robustness to various specifications such as the number of unit roots in the system would be an interesting topic for future research.

[27]Based on the continuity of the finite sample distribution of least-squares estimator, applied macroeconomists may ignore explosive roots with magnitudes artibrarily close to unity. Hence depending on the objective of the analysis, ignoring *slightly* explosive roots may be another alternative. However, it is not clear as to what would be a reasonable cutoff for categorizing an autoregressive root as *slightly* explosive. Moreover, such a cutoff would vary with the system and the purpose of the research.

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Results

Table 1: Explosive Roots in Estimated Level VAR Models

Explosive Roots (percent)	CEE 99	CEE 05	EE 95
VAR(p)	46.4	47.7	41.9
VAR(p) - Pope	100	100	100
VAR(p) - Kilian	77.2	92.4	85.3

DGP: VECM(p*) with cointegration rank h. **Estimated Model: VAR(p)**. p^* and p equal 4 for CEE (1999, 2005) and 6 for EE (1995). h equals 5, 4 and 2 for CEE(1999), CEE (2005) and EE(1995) respectively.

Table 2: Sensitivity Analysis of Cointegration Rank h in the DGP

Explosive Roots (percent)	CEE 99				CEE 05				EE 95			
h	3	4	*5*	n-1	3	*4*	5	n-1	1	*2*	3	n-1
VAR(p)	41.7	44.5	*46.4*	36.7	48.7	*47.7*	39.9	28.4	45.2	*41.9*	39.9	35.7
VAR(p) - Pope	100	100	*100*	100	100	*100*	100	100	100	*100*	100	100
VAR(p) - Kilian	89.4	83.7	*77.2*	67.9	96.5	*92.4*	88.9	62.4	89.1	*85.3*	75.4	59.4

DGP: VECM(p*) with cointegration rank h. **Estimated Model: VAR(p)**. p^* and p equal 4 for CEE (1999, 2005) and 6 for EE (1995). Benchmark cases from Table 1 are in *italics*. n is the number of variables in the system: n equals 7, 9 and 5 for CEE-(1999), CEE (2005) and EE(1995) respectively.

Table 3: Explosive Roots in Estimated Level VAR Models - **DGP: VECM**

Explosive Roots (percent)	CEE 99			CEE 05			EE 95		
p	3	*4*	5	3	*4*	5	5	*6*	7
FULL SAMPLE									
VAR(p)	44.7	*46.4*	47.1	46.3	*47.7*	50.3	42.4	*41.9*	42.5
VAR(p) - Pope	100	*100*	100	100	*100*	100	100	*100*	100
VAR(p) - Kilian	75.0	*77.2*	77.0	95.3	*92.4*	93.3	83.7	*85.3*	85.5
p	2	3	*4*	2	3	*4*	4	5	*6*
POST-BRETTON WOODS									
VAR(p)	47.3	47.9	49.7	46.9	50.4	55.9	42.4	43.9	44.0
VAR(p) - Pope	100	100	100	100	100	100	100	100	100
VAR(p) - Kilian	79.3	77.7	79.5	92.2	92.8	93.2	76.9	83.2	84.2
VOLCKER & POST-VOLCKER									
VAR(p)	48.8	50.3	56.1	48.2	56.5	70.2	42.6	44.7	44.9
VAR(p) - Pope	100	100	100	100	100	100	100	100	100
VAR(p) - Kilian	82.6	81.5	82.5	92.4	91.5	94.2	82.9	85.1	87.1

DGP: VECM(p*) with cointegration rank h. **Estimated Model: VAR(p)**. p^* equals 4 for CEE (1999, 2005) and 6 for EE (1995). Benchmark cases from Table 1 are in *italics*. h equals 5, 4 and 2 for CEE(1999), CEE (2005) and EE(1995) respectively.

Table 4: Explosive Roots in Estimated Level VAR Models - **DGP: VAR**

Explosive Roots (percent)	CEE 99			CEE 05			EE 95		
p	3	*4*	5	3	*4*	5	5	*6*	7
FULL SAMPLE									
VAR(p)	24.9	*25.9*	29.6	12.4	*12.9*	16.8	21.4	*19.6*	17.7
VAR(p) - Pope	95.5	*95.2*	95.0	82.3	*79.4*	80.3	92.0	*88.6*	87.0
VAR(p) - Kilian	74.1	*72.8*	72.6	64.0	*64.2*	64.3	56.4	*53.3*	51.3
p	2	3	*4*	2	3	*4*	4	5	*6*
POST-BRETTON WOODS									
VAR(p)	33.0	35.1	41.3	24.2	26.9	35.9	24.1	22.5	20.4
VAR(p) - Pope	98.9	98.5	98.5	95.3	92.7	93.5	94.5	92.5	90.3
VAR(p) - Kilian	80.8	81.7	81.4	82.3	85.3	87.2	55.4	59.7	60.2
VOLCKER & POST-VOLCKER									
VAR(p)	41.5	46.3	56.0	33.6	44.1	61.4	30.9	31.6	31.7
VAR(p) - Pope	99.6	99.5	99.5	97.5	97.1	98.1	97.3	96.5	96.1
VAR(p) - Kilian	83.8	86.4	89.5	88.1	92.0	93.3	70.0	75.4	78.4

DGP: VAR(p*). **Estimated Model: VAR(p)**. p^* equals 4 for CEE (1999, 2005) and 6 for EE (1995).

Table 5: Explosive Roots in Estimated VECM - **DGP: VECM**

Explosive Roots (percent)	CEE 99			CEE 05			EE 95		
p	3	*4*	5	3	*4*	5	5	*6*	7
Full Sample									
VECM(p)	1.8	*2.3*	2.3	0.0	*0.6*	0.2	0.1	*0.4*	0.3
p	2	3	*4*	2	3	*4*	4	5	*6*
Post-Bretton Woods									
VECM(p)	4.4	3.4	4.8	0.2	0.2	0.8	0.3	0.8	2.5
Volcker & Post-Volcker									
VECM(p)	5.6	6.6	9.4	0.4	1.0	5.6	0.4	1.1	3.2

DGP: VECM(p*) with cointegration rank h. **Estimated Model: VECM(p)** with cointegration rank h. p^* equals 4 for CEE (1999, 2005) and 6 for EE (1995). Benchmark cases are in *italics*. h equals 5, 4 and 2 for CEE(1999), CEE (2005) and EE(1995) respectively.

Table 6: Explosive Roots in Estimated VECMs - **DGP: VAR**

Explosive Roots (percent)	CEE 99			CEE 05			EE 95		
p	3	*4*	5	3	*4*	5	5	*6*	7
Full Sample									
VECM(p)	0.2	*0.9*	1.1	0.1	*0.0*	0.1	0.0	*0.1*	0.1
p	2	3	*4*	2	3	*4*	4	5	*6*
Post-Bretton Woods									
VECM(p)	1.0	1.3	3.1	0.0	0.1	0.2	0.3	0.5	1.1
Volcker & Post-Volcker									
VECM(p)	2.0	3.4	8.5	1.2	5.0	3.9	0.9	0.8	1.3

DGP: VAR(p*). **Estimated Model: VECM(p)** with cointegration rank h. p^* equals 4 for CEE (1999, 2005) and 6 for EE (1995). h equals 5, 4 and 2 for CEE(1999), CEE (2005) and EE(1995) respectively. Benchmark cases are in *italics*.

Table 7: Sensitivity to Cointegration Rank in Estimated VECMs

Cointergration Rank	EE 95	CEE 99	CEE 05
1	0.0	0.4	0.0
2	0.2	0.6	0.0
3	1.2	1.2	0.0
4	9.5	2.1	0.0
5	41.9	2.3	1.2
6	-	11.2	2.0
7	-	46.4	5.1
8	-	-	12.4
9	-	-	47.7

DGP: VECM(p) with cointegration ranks of 5, 4 and 2 for CEE(1999), CEE (2005) and EE(1995) respectively. **Estimated Model: VECM(p)**. p equals 4 for CEE (1999, 2005) and 6 for EE (1995). h equals 5, 4 and

Appendix

I. Data Description

This appendix describes the datasets in CEE (1999), CEE (2005) and EE (1995).

CEE (1999)

Sample Period: 1964:2 to1995:2 (quarterly)

Variables: (i) real output(Y), (ii) implicit GDP deflator (P_{def}), (iii) change in commodity prices (P_{com}), (iv), federal funds rate (FFR), (v) nonborrowed reserves (NBR), (vi) total reserves ($TOTR$), and (vii) $M1$.

Notes:

1. All variables except for P_{com} and FFR are in 100*log form
2. CEE (1999) report results with both $M1$ and $M2$ in their benchmark analyses. This paper only presents results with $M1$. Results are essentially the same if $M1$ is replaced by $M2$.

CEE (2005)

Sample Period: 1964:2 to1995:2 (quarterly)

Variables: (i) real output(Y), (ii) inflation(Inf), (iii) consumption (C), (iv) investment (I) (v) real wage (w), (vi) productivity($Prod$), (vii) federal funds rate (FFR), (viii) money growth for $M2$ ($M2_{growth}$), and (ix) real profits (π).

Notes:

1. Inf and $M2_{growth}$ are calculated as the difference of 100*log of price level and $M2$ respectively. The remaining variables (except for FFR) are in 100*log form

EE (1995)

Sample Period: 1970:1 to1991:12 (monthly)

Variables: (i) US industrial production(Y_{ind}), (ii) consumer price level (CPI), (iii) ratio of nonborrowed to total reserves ($NBRX$), (iv) a measure of the difference between US and French interest rate($R_{US} - R_{France}$), and (v) monthly average of spot \$/Franc nominal exchange rate ($e_{\$/Franc}$).

Notes:

1. EE (1995) estimate level VAR models with five, seven and eight variables and examine five different nominal and real exchange rates. In the interest of brevity, this paper only presents results for their $e_{\$/Franc}$ model with five variables. Results for other real or nominal exchange rates are essentially the same. Results with their seven variable VAR models are also very similar.

II. Unit Root Tests

Unit Root Tests

CEE(1999)	SDF	PP	MPP	MSB	PT	MPT	DF-GLS	MPP-GLS
Y	(0.82)	1.37	1.47	1.42	189.3	146.6	1.52	1.38
P_{def}	(1.32)	(10.93)*	(10.85)*	0.20	3.52	2.80*	(0.55)	(1.27)
P_{com}	(3.38)*	(21.32)*	(19.48)*	(0.16)*	(1.39)*	(1.39)*	(3.37)*	(19.48)*
FFR	(2.24)	(3.11)	(5.97)	0.29	4.82	4.13	(1.57)	(5.17)
NBR	(0.49)	1.61	1.70	1.06	106.9	88.21	1.12	1.24
$TOTR$	(0.59)	1.29	1.38	0.82	64.80	53.00	0.51	1.24
$M1$	(1.08)	0.08	0.17	0.54	27.40	22.06	0.08	(1.32)

CEE(2005)	SDF	PP	MPP	MSB	PT	MPT	DF-GLS	MPP-GLS
Y	(0.82)	1.37	1.47	1.42	189.3	146.6	1.52	1.38
Inf	(1.57)	(4.17)	(3.31)	0.39	8.63	7.39	(1.30)	(2.79)
C	(0.52)	1.19	1.29	1.20	134.46	102.97	1.41	1.02
I	(1.46)	0.18	0.28	0.56	30.02	23.43	0.14	0.18
w	(2.08)	(2.54)	(2.48)	0.44	12.50	9.76	(0.95)	(2.53)
$Prod$	(0.65)	1.48	1.60	1.32	170.02	130.57	1.92	1.44
FFR	(2.24)	(3.11)	(5.97)	0.29	4.82	4.13	(1.57)	(5.17)
$M2_{growth}$	(0.73)	(6.75)	2.51	0.43	9.43	9.57	(0.82)	(2.46)
π	(1.25)	0.49	0.56	1.05	86.92	69.72	0.54	0.49

EE(1995)	SDF	PP	MPP	MSB	PT	MPT	DF-GLS	MPP-GLS
Y_{ind}	(1.53)	0.85	0.87	1.10	93.61	80.63	0.85	0.86
CPI	(1.38)	(2.90)	(2.87)	0.34	9.25	8.03	(0.18)	(0.72)
$NBRX$	(2.51)	(14.00)*	(13.02)*	0.19*	1.96*	1.97*	(2.49)*	(13.25)*
$R_{US} - R_{France}$	(2.21)	(9.64)*	(9.07)*	0.22*	3.13*	3.13*	(2.11)*	(9.88)*
$e_{\$/Franc}$	(1.55)	(5.15)	(5.14)	0.31	4.73	4.77	(1.55)	(5.14)

* denotes the rejection of the hypothesis at 5%significance level. SDF, PP, MPP, MSB, PT, MPT, DF-GLS, MPP-GLS denote Said-Dickey-Fuller, Phillips-Perron, Modified-Phillips-Perron, Modified-Sargan Bhargava, ERS feasibale point test, Modified feasible point test, DF with GLS detrending and Modified Philips-Perrron with GLS detrending respectively.

III. Cointegration Rank Tests

Johansen's Tests

Eigenvalue CEE(1999)	Eig_{max}	$Trace$	Rank (h)	Eig_{max}(5% c.v)	$Trace$(5% c.v)
0.4388	69.9**	217.1**	0	45.28	124.24
0.3455	51.3**	147.2**	1	39.37	94.15
0.2653	37.3*	95.9**	2	33.46	68.52
0.2066	28.0*	58.6**	3	27.07	47.21
0.1560	20.53	30.6*	4	20.97	29.68
0.0540	6.72	10.09	5	14.07	15.41
0.0275	3.37	3.37	6	3.76	3.76
CEE(2005)					
0.4445	71.12**	251.72**	0	57.12	192.89
0.3274	47.99	180.59**	1	51.42	156.00
0.2480	34.49	132.60*	2	45.28	124.24
0.2244	30.75	98.12*	3	39.37	94.15
0.2040	27.61	67.37	4	33.46	68.52
0.1266	16.38	39.76	5	27.07	47.21
0.1009	12.68	23.37	6	20.97	29.68
0.0451	5.59	10.51	7	14.07	15.41
0.0399	4.92	4.92	8	3.76	3.76
EE(1995)					
0.1453	38.63*	87.08**	0	33.46	68.52
0.0827	21.22	48.45*	1	27.07	47.21
0.0696	17.75	27.23	2	20.97	29.68
0.0274	6.83	9.48	3	14.07	15.41
0.0107	2.65	2.65	4	3.76	3.76

*(**) denotes the rejection of the hypothesis at 5%(1%) significance level. Testing Assumption: Linear trend in data

IV. ESTIMATION PROCEDURES

This section briefly describes the estimation and bias correction procedures used in this paper.

A. VECM ESTIMATION

Reduced form VECM(p) with a cointegration rank of *h* can be written in the form:

$$\Delta Y_t = c + \zeta_1 \Delta Y_{t-1} + \zeta_2 \Delta Y_{t-2} + \ ..... \ + \zeta_3 \Delta Y_{t-p} + \zeta_0 Y_{t-1} + \varepsilon_t \qquad (1)$$

$$with \quad E(\varepsilon_t) = 0, \quad E(\varepsilon_t \varepsilon_\tau ') = \Omega \ \text{ for } t = \tau; \quad 0 \text{ otherwise.}$$

Assuming normal errors, VECM coefficients can be estimated by maximizing the following likelihood function:

$$L(\Omega, \zeta_1, ..., \zeta_{p-1}, c, \zeta_0) = (-Tn/2)\log(2\pi) - (T/2)\log|\Omega| \qquad (2)$$

$$-1/2 \sum_{t=1}^{T} \begin{matrix} [(\Delta Y_t - c - \zeta_1 \Delta Y_{t-1} - ...... - \zeta_{p-1} \Delta Y_{t-p+1} - \zeta_0 Y_{t-1})' \\ \Omega^{-1} (\Delta Y_t - c - \zeta_1 \Delta Y_{t-1} - ...... - \zeta_{p-1} \Delta Y_{t-p+1} - \zeta_0 Y_{t-1})] \end{matrix}$$

subject to $\zeta_0 = -BA'$

where *B* is an (*n* x *h*) matrix of adjustment rates, *A'* is an (*h* x *n*) matrix of cointegrating vectors, *n* is the number of variables in the system and *h* is the cointegration rank based on Johansen's cointegration rank test.

The maximum likelihood estimates in (2) can be obtained by implementing Johansen's algorithm, which is summarized in the following steps. For a more detailed description and proofs, see Hamilton (1994).

- Calculate the following Auxiliary Regressions

$$\Delta Y_t = \pi_0 + \Pi_1 \Delta Y_{t-1} + \Pi_2 \Delta Y_{t-2} + \ ..... \ + \Pi_{p-1} \Delta Y_{t-p+1} + u_t \qquad (3)$$

$$Y_{t-1} = \theta + \chi_1 \Delta Y_{t-1} + \chi_2 \Delta Y_{t-2} + \ ..... \ + \chi_{p-1} \Delta Y_{t-p+1} + v_t \qquad (4)$$

where these vector regressions are estimated by OLS (equation by equation)

- Calculate the Canonical Correlations

  Next, calculate the following sample covariance matrices of the OLS residuals, u_t and v_t

  $$\Sigma_{vv} = (1/T)\sum_{t=1}^{T} v_t v_t'$$

  $$\Sigma_{uu} = (1/T)\sum_{t=1}^{T} u_t u_t'$$

  $$\Sigma_{vu} = \Sigma_{uv}(1/T)\sum_{t=1}^{T} v_t u_t'$$

  Subsequently, eigenvalues of the matrix $\Sigma_{vv}{}^{-1}\Sigma_{vu}\Sigma_{uu}{}^{-1}\Sigma_{uv}$ are computed with the eigenvalues ordered $\lambda_1 > \lambda_2 > \ldots > \lambda_n$

- Calculate Maximum Likelihood Estimates of Parameters

  The *h* normalized eigenvectors corresponding to the *h* largest eigenvalues are collected in the following matrix:

  $$A = [\, a_1 \;\, a_2 \;\, \ldots\ldots \;\, a_h \,]$$

  Johansen suggested normalizing the eigenvectors so that $a_i{}'\Sigma_{vv}a_i = 1$ for *i =1*, .. , *h*

  Then the maximum likelihood estimates of ζ_0 and ζ_i are

  $$\zeta_0 = \Sigma_{uv} AA'$$

  $$\zeta_i = \Pi_i - \zeta_0 \chi_i \text{ for } i = 1, \ldots, p\text{-}1$$

  Next, the maximum likelihood estimates of α and Ω are give by:

  $$\alpha = \pi_0 - \zeta_0 \theta$$

  $$\Omega = (1/T)\sum_{t=1}^{T} \left[(u_t - \zeta_0 v_t)(u_t - \zeta_0 v_t)' \right]$$

B. Pope Bias Correction

VAR coefficients after bias correction based on Pope's expression are estimated by implementing the following steps:

- Estimate a VAR(p) model without a constant are convert it to VAR(1) companion form. For instance, a VAR(p)

$$Y_t = \Phi_1 Y_{t-1} + \Phi_2 Y_{t-2} + \; ..... \; + \Phi_p Y_{t-p} + \varepsilon_t \tag{5}$$

can be rewritten as:

$$\xi_t = F\xi_{t-1} + v_t \tag{6}$$

$$E(v_t v_\tau ') = \begin{Bmatrix} Q & for\ t = \tau \\ 0 & otherwise \end{Bmatrix}$$

$$\underset{np\ x\ np}{Q} = \begin{pmatrix} \Omega & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$where \quad \underset{(np\, x 1)}{\xi_t} = \begin{pmatrix} Y_t \\ Y_{t-1} \\ \vdots \\ Y_{t-p+1} \end{pmatrix}, \quad \underset{(np\, x 1)}{v_t} = \begin{pmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\& \quad \underset{(np\ x\ np)}{F} = \begin{pmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_{p-1} & \Phi_p \\ I_n & 0 & \cdots & 0 & 0 \\ 0 & I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & I_n & 0 \end{pmatrix}$$

- Next calculate Pope's expression for the bias in demeaned VAR(1)

$$B_T = -\frac{b}{T} + O(T^{-3/2})$$

where b is given by:

$$b = Q[(I - F')^{-1} + F'\{I - (F')^2\}^{-1} + \sum \lambda (I - \lambda F')^{-1}]\Gamma(0)^{-1} \tag{7}$$

The sum is over the eigenvalues λ of F, weighted by their multiplicities. Q denotes the conditional variance of v_t and $\Gamma(0)$ denotes the variance of ξ.

Hence,

$$\begin{aligned}\Gamma(0) &= E(\xi_t \xi_t') \\ &= E\left[\left(F\xi_{t-1} + v_t\right)\left(F\xi_{t-1} + v_t\right)'\right] = FE(\xi_{t-1}\xi_{t-1}')F' + E(v_t v_t')\end{aligned} \tag{8}$$

$$\begin{aligned}&or \\ &\Gamma(0) = F\Gamma(0)F' + Q \\ &vec(\Gamma(0)) = (F \otimes F).vec(\Gamma(0)) + vec(Q), \\ &vec(\Gamma(0)) = [I_r - (F \otimes F)]^{-1} vec(Q)\end{aligned}$$

where $\otimes$ denotes the Kronecker product and $r = n \times p$.

- Finally Pope's bias is subtracted from the estimated VAR coefficients to yield the bias corrected coefficients.

C. Kilian Bias Correction

Kilian (1998) suggests the following algorithm for the bias-corrected bootstrap method:

- Estimate the following VAR(p) and generate 1000 bootstrap replications $\hat{\phi}^*$ from

$$Y_t = \hat{c} + \hat{\Phi}_1 Y_{t-1} + \hat{\Phi}_2 Y_{t-2} + \ \ldots\ldots \ + \hat{\Phi}_p Y_{t-p} + \varepsilon_t \tag{9}$$

  using standard nonparametric bootstrap techniques.

- Approximate the bias term $\Psi = E(\hat{\phi} - \phi)$ by $\Psi^* = E^*(\hat{\phi}^* - \hat{\phi})$, which suggests the bias estimate $\hat{\Psi} = \overline{\phi}^* - \hat{\phi}$, where $\overline{\phi}^*$ is the mean of the bootstrap sample of $\hat{\phi}^*$.
- Bias-corrected coefficient estimate, $\tilde{\phi}^*$ is given by $\tilde{\phi}^* = \hat{\phi} - \hat{\psi}$.

Note: Subsequently, Kilian also implements a stationarity correction to avoid pushing stationary estimates into the non-stationary region. For the reasons discussed in the paper, I do not implement Kilian's stationarity correction.

D. ESTIMATING FREQUENCY OF EXPLOSIVE ROOTS

First a VAR(p) process is written in the VAR(1) form as in (6). Subsequently, the stability of VAR(p) is checked by calculating the absolute eigenvalues of the matrix *F*, which correspond to the autoregressive roots of the system. If $\lambda_{\max}$, the modulus of the largest root of an estimated VAR model lies outside the unit circle in a given sample, the VAR model would be unstable for that Monte Carlo sample. Frequency of explosive roots corresponds to the proportion of unstable VAR draws in the Monte Carlo samples. In order to allow for rounding off errors, I consider a VAR model to be explosive only if its _max exceeds a threshold value of 1.00001 (instead of exactly one). Results are essentially the same for other thresholds such as 1.0001 or 1.0005.